

A SYSTEM OF POLYNOMIAL EQUATIONS RELATED TO THE JACOBIAN CONJECTURE

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ABSTRACT. We prove that the Jacobian conjecture is false if and only if there exists a solution to a certain system of polynomial equations. We analyse the solution set of this system. In particular we prove that it is zero dimensional.

Introduction

Let K be a characteristic zero field. The Jacobian Conjecture (JC) in dimension two, stated by Keller in [6], says that any pair of polynomials $P, Q \in R := K[x, y]$ with

$$[P, Q] := \partial_x P \partial_y Q - \partial_x Q \partial_y P \in K^\times$$

defines an automorphism of R .

T. T. Moh analyses in [7] the existence of possible counterexamples (P, Q) with total degree of P and Q lower than 101 and finds four exceptional cases $(m, n) = (48, 64)$, $(m, n) = (50, 75)$, $(m, n) = (56, 84)$ or $(m, n) = (66, 99)$, where $(n, m) = (\deg(P), \deg(Q))$. Then he discards these cases by hand solving certain Ad Hoc systems of equations for the coefficients of the possible counterexamples. Motivated by this we introduce and begin the study of a polynomial system $S_t(n, m, (\lambda_i), F_{1-n})$ of $m + n - 2$ equations with coefficients in a commutative K -algebra D and $m + n - 2$ variables. Here $(\lambda_i)_{0 \leq i \leq m+n-2}$ is a family of $m + n - 2$ elements of K and $F_{1-n} \in D$. Among other results, we prove that a particular instance of this system (with $D = K[y]$ and $F_{1-n} = y$) has a solution in D^{m+n-2} if and only if there exists a counterexample (P, Q) to JC with $(n, m) = (\deg(P), \deg(Q))$. For this we use an equivalent formulation of the JC due to Abhyankar [1], which asserts that JC is true if for all Jacobian pairs (P, Q) either $\deg(P)$ divides $\deg(Q)$ or viceversa. We also prove that if D is an integral domain, then the set of solutions of $S_t(n, m, (\lambda_i), F_{1-n})$ is finite. After that, we analyse the case in which $\lambda_i = 0$ for $i > 0$, which we call the homogeneous system, giving a very detailed description of its solutions. In Proposition 4.3 we show that the homogeneous system has always a solution, using a result of [9].

Our system provides a significative reduction of the number of equations and variables needed in order to verify the existence of a counterexample to JC at (n, m) , where the most naive approach needs $m(m+1)/2 + n(n+1)/2$ variables and $(m+n-1)(m+n-2)/2$ equations. However the number of equations is still too big to have a realistic chance to verify the existence of a counterexample to JC for the pairs $(m, n) = (48, 64)$, $(m, n) = (50, 75)$, $(m, n) = (56, 84)$ or $(m, n) = (66, 99)$, which are the cases found in [7].

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In the last section we show how one has to proceed in a concrete example, analysing the case $(n, m) = (50, 75)$. Using a reduction of degree technique as in Section 8 of [3], one can show that in that case there must exist a pair $(P, Q) \in K[x, y, y^{-1}]$ with $(\deg_x(P), \deg_x(Q)) = (4, 6)$ or $(\deg_x(P), \deg_x(Q)) = (6, 9)$, satisfying certain additional properties. Among others, the Jacobian $[P, Q] \notin K^\times$. Due to this fact we must use a slight variation of the system $S_t(n, m, (\lambda_i), F_{1-n})$. Our computations provide an independent verification of Moh's result: There is no counterexample at $(50, 75)$. An advantage of our system of equations compared to the ones used by Moh, is its form, which is canonical even for the modified systems. On one hand this allows to program more general algorithms in order to verify concrete cases, following the procedure suggested in Section 5. On the other hand, further analysis of the structure of the system of equations could give some progress in solving the JC, discarding at least some infinite families of possible counterexamples, and not only single cases.

1 The Jacobian Conjecture as a system of equations

Let K be a characteristic zero field and let D an arbitrary commutative K -algebra. In this section we introduce a polynomial system $S_t(n, m, (\lambda_i), F_{1-n})$ of $m + n - 2$ equations with $m + n - 2$ variables, where $(\lambda_i)_{0 \leq i \leq m+n-2}$ is a family of $m + n - 2$ elements of K and $F_{1-n} \in D$. The main results are Theorem 1.9 and Corollary 1.17, in which we show that there exists a counterexample (P, Q) to JC with $(\deg(P), \deg(Q)) = (m, n)$ if and only if $S_t(n, m, (\lambda_i), y)$ has a solution in $K[y]^{m+n-2}$ for some $\lambda_1, \dots, \lambda_{m+n-2} \in K$.

A non-zero element $\mathbf{w} := (w_1, w_2) \in \mathbb{Z}^2$ is called a *direction* if $\gcd(w_1, w_2) = 1$ and $w_1 > 0$ or $w_2 > 0$. In the sequel for each direction $\mathbf{w} := (w_1, w_2)$, we write $|\mathbf{w}| := w_1 + w_2$. Furthermore, by the sake of simplicity we set $R := K[x, y]$. A polynomial $P \in R$ is said to have a *Jacobian mate* $Q \in R$ if

$$[P, Q] := \partial_x P \partial_y Q - \partial_y P \partial_x Q \in K^\times.$$

In this case P and Q are called *Jacobian polynomials* and (P, Q) is called a *Jacobian pair*.

To each direction \mathbf{w} we associate the so-called \mathbf{w} -grading on R ,

$$R := \bigoplus_{d \in \mathbb{Z}} R_d(\mathbf{w}),$$

where $R_d(\mathbf{w})$ is the K -vector subspace of R generated by all monomials $x^i y^j$ such that $i w_1 + j w_2 = d$. If there is no confusion possible, we will write R_d instead of $R_d(\mathbf{w})$. For $P \in R \setminus \{0\}$ we denote by P_+ the \mathbf{w} -homogeneous part of P of highest degree. Furthermore, if $P_+ \in R_d(\mathbf{w})$, then we say that the \mathbf{w} -degree of P is d , and write $\mathbf{wdeg}(P) = d$. For convenience we set $\mathbf{wdeg}(0) = -\infty$. As usual we will write $\deg(P)$, $\deg_x(P)$ and $\deg_y(P)$ instead of $(1, 1)\deg(P)$, $(1, 0)\deg(P)$ and $(0, 1)\deg(P)$, respectively. We also say that P is homogeneous if it is $(1, 1)$ -homogeneous. We have the following result due to Abhyankar:

Proposition 1.1 ([8, Theorem 10.2.23]). *The Jacobian conjecture is false if and only if there exists a Jacobian pair (P, Q) , such that neither $\deg(P)$ divides $\deg(Q)$ nor $\deg(Q)$ divides $\deg(P)$.*

Remark 1.2. The arguments in the proof of the above proposition show that if (P, Q) is a Jacobian pair such that neither $\deg(P)$ divides $\deg(Q)$ nor $\deg(Q)$ divides $\deg(P)$, then (P, Q) is a counterexample to JC.

We will use freely that if φ is an automorphism of R , then

$$[\varphi(P), \varphi(Q)] = \varphi([P, Q])[\varphi(x), \varphi(y)].$$

Let (P, Q) be as in Proposition 1.1. For each $\lambda \in K$ we define $\varphi_\lambda \in \text{Aut}(R)$ by

$$\varphi_\lambda(x) := x \quad \text{and} \quad \varphi_\lambda(y) := y + \lambda x.$$

Let $n := \deg(P)$, $m := \deg(Q)$ and $\mathbf{w} := (1, 0)$. It is easy to check that there exists $\lambda \in K$ such that $\varphi_\lambda(P)_+ = \mu_P x^n$ and $\varphi_\lambda(Q)_+ = \mu_Q x^m$, with $\mu_P, \mu_Q \in K^\times$. Consequently, since φ_λ is $(1, 1)$ -homogeneous,

$$\varphi_\lambda(P) = \mu_P x^n + \mu_{n-1} x^{n-1} + \cdots + \mu_0,$$

with $\mu_{n-i} \in K[y]$ and $\deg(\mu_{n-i}) \leq i$. Let ϕ be the automorphism of R defined by $\phi(y) := y$ and $\phi(x) := x - \frac{\mu_{n-1}}{\mu_P}$. Replacing P and Q by $\frac{1}{\mu_P} \phi(\varphi_\lambda(P))$ and $\frac{1}{\mu_Q} \phi(\varphi_\lambda(Q))$, respectively, we can assume without loss of generality that

$$P = x^n + \gamma_{n-2} x^{n-2} + \cdots + \gamma_0 \quad \text{and} \quad Q = x^m + \delta_{m-1} x^{m-1} + \cdots + \delta_0, \quad (1.1)$$

with $\gamma_{n-i}, \delta_{m-i} \in K[y]$ and $\deg(\gamma_{n-i}), \deg(\delta_{m-i}) \leq i$. Furthermore, a standard straightforward computation shows that there exists a unique $C \in K[y]((x^{-1}))$ such that

$$C^n = P \quad \text{and} \quad C = x + C_0 + C_{-1} x^{-1} + C_{-2} x^{-2} + \cdots, \quad (1.2)$$

where $C_k \in K[y]$, $C_0 = 0$ and $\deg_y(C_k) \leq -k + 1$ for all $k \leq -1$. It is easy to see that C is invertible and

$$C^j = x^j + (C^j)_{j-1} x^{j-1} + (C^j)_{j-2} x^{j-2} + (C^j)_{j-3} x^{j-3} + (C^j)_{j-4} x^{j-4} + \cdots \quad \text{for all } j \in \mathbb{Z},$$

where $(C^j)_{-k} \in K[y]$, $(C^j)_{j-1} = 0$ and $\deg_y((C^j)_k) \leq -k + j$ for all $k \leq j - 2$.

Definition 1.3. Let $H = \sum a_{ij} x^i y^j \in K[y]((x^{-1})) \setminus \{0\}$. The *support* of H is

$$\text{Supp}(H) := \{(i, j) \in \mathbb{Z} \times \mathbb{N}_0 : a_{ij} \neq 0\}.$$

Let $\mathbf{w} = (w_1, w_2)$ be a direction. For $H \in K[y]((x^{-1})) \setminus \{0\}$, we write

$$\mathbf{wdeg}(H) := \sup\{i w_1 + j w_2 : (i, j) \in \text{Supp}(H)\}.$$

Of course it is possible that $\mathbf{wdeg}(H) = +\infty$.

For $P, Q \in K[y]((x^{-1}))$ we define

$$[P, Q] := \partial_x P \partial_y Q - \partial_y P \partial_x Q,$$

where $\partial_x P$ denotes the formal derivative of P with respect to x , etcetera. It is easy to see that

$$\mathbf{wdeg}([P, Q]) \leq \mathbf{wdeg}(P) + \mathbf{wdeg}(Q) - |\mathbf{w}|,$$

for any direction \mathbf{w} .

Definition 1.4 ([8, page 247]). Let P be a polynomial of degree > 1 having a Jacobian mate of degree > 1 and let \mathbf{w} be a direction. Let $R[P_+^{-1}]$ be the localization of R in P_+ . The ring extension \tilde{R}_{P_+} of R is the set of formal sums $f := \sum_{i \in \mathbb{Z}} f_i$, where each f_i is a \mathbf{w} -homogeneous element of $R[P_+^{-1}]$ of degree i and $f_i = 0$ for $i \gg 0$. If $f \neq 0$, then the highest i with $f_i \neq 0$, denoted by $\mathbf{wdeg}(f)$, is called the \mathbf{w} -degree of f , while f_i is denoted by f_+ .

Proposition 1.5. If $\mathbf{w} = (1, 1)$ and P is as in (1.1), then \tilde{R}_{P_+} is in a natural way a graded subalgebra of $K[y]((x^{-1}))$.

Proof. Write

$$P_+ = x^n + \alpha_1 y x^{n-1} + \alpha_2 y^2 x^{n-2} + \cdots + \alpha_n y^n = x^n - B$$

where $\alpha_1, \dots, \alpha_n \in K$ and $B := -\alpha_1 y x^{n-1} - \alpha_2 y^2 x^{n-2} - \cdots - \alpha_n y^n$ (actually $\alpha_1 = 0$ but we do not use this fact). A direct computation shows that P_+ is invertible in $K[y]((x^{-1}))$ and that

$$P_+^{-1} = x^{-n} + x^{-2n} B + x^{-3n} B^2 + x^{-4n} B^3 + \cdots.$$

Note that the sum in the right side of this equality is well defined since

$$\deg_x(x^{-in-n} B^i) \leq (n-1)i - in - n = -n - i.$$

In order to finish the proof it suffices to show that each series

$$\sum_{i \leq r} f_i \quad \text{with } f_i \in K[y]((x^{-1})) \text{ such that } \deg(f_i) = i,$$

is summable in $K[y]((x^{-1}))$. But this follows from the fact that $\deg(f_i) = i$ implies that

$$f_i = \beta_0 x^i + \beta_1 x^{i-1} + \beta_2 x^{i-2} + \dots$$

with $\beta_i \in K[y]$ and $\deg(\beta_i) \leq i$. \square

In order to prove Theorem 1.9, we will need to use the following result, in which P_+ and F_+ are taken with respect to the $(1, 0)$ -grading.

Lemma 1.6. *Let $P, F \in K[y]((x^{-1}))$ be such that $P_+ = x^n$, $\deg_x(F) \leq 1 - n$ and $[P, F] \in K^\times$. Then $F_+ = (\mu_0 + \mu_1 y)x^{1-n}$ with $\mu_1 \neq 0$.*

Proof. Let $P = \sum_{i \leq n} P_i$ and $F = \sum_{j \leq 1-n} F_j$ be the $(1, 0)$ -homogeneous decompositions of P and F . Then the $(1, 0)$ -homogeneous decomposition of

$$[P, F] = [P, F]_0 + [P, F]_{-1} + [P, F]_{-2} + \dots$$

is given by

$$[P, F]_k = \sum_{i+j=k+1} [P_i, F_j].$$

Write $F_{1-n} = x^{1-n} f_{1-n}(y)$. Since $[P, F] \in K^\times$, we have

$$n f'(y) = [x^n, x^{1-n} f_{1-n}(y)] = [P_n, F_{1-n}] = [P, F]_0 = [P, F] \in K^\times$$

So $f'(y) \in K^\times$, which implies that $f(y) = \mu_0 + \mu_1 y$ for some $\mu_0 \in K$ and $\mu_1 \in K^\times$, as desired. \square

We also will need the following particular case of [8, Lemma 10.2.11]:

Proposition 1.7. *Let $\mathbf{w} = (1, 1)$ and let P be as in (1.1) and $C \in K[y]((x^{-1}))$ as in (1.2). Assume P has a Jacobian mate $Q \in R$ of degree > 1 and let $\tilde{Q} \in \tilde{R}_{P_+}$ be such that $[P, \tilde{Q}] \in K^\times$. If*

$$\deg(P) + \deg(\tilde{Q}) - 2 > 0,$$

then there exists $j \in \mathbb{Z}$ and $\lambda \in K^\times$ such that $C^j \in \tilde{R}_{P_+}$ and $\deg(\tilde{Q} - \lambda C^j) < \deg(\tilde{Q})$.

Remark 1.8. The number n that appears in the statement of [8, Lemma 10.2.11] is not the degree of P , but only a divisor of $\deg(P)$. The element $P^{\frac{1}{n}}$, introduced in [8] above of Lemma 10.2.10, equals $\mu C^{\deg(P)/n}$ where $\mu \in K^\times$ and n is as in [8, Lemma 10.2.11].

Theorem 1.9. *The JC is false if and only if there exist*

- $P, Q \in R$ and $C, F \in K[y]((x^{-1}))$,
- $n, m \in \mathbb{N}$ such that $n \nmid m$ and $m \nmid n$,
- $\lambda_i \in K$ ($i = 0, \dots, m+n-2$) with $\lambda_0 = 1$,

such that

- C has the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \dots \quad \text{with each } C_{-i} \in K[y],$$

- $\deg(C) = 1$ and $\deg(F) = 2 - n$,
- $F_+ = x^{1-n}y$, where F_+ is taken with respect to the $(1, 0)$ -grading,
- $C^n = P$ and $Q = \sum_{i=0}^{m+n-2} \lambda_i C^{m-i} + F$.

Furthermore, under these conditions, (P, Q) is a counterexample to the Jacobian conjecture.

Proof. \Rightarrow) By Proposition 1.1 we know that there exists a Jacobian pair (P, Q) that is a counterexample, such that neither $n \nmid m$ nor $m \nmid n$, where $n := \deg(P)$ and $m := \deg(Q)$. Furthermore, by the discussion below that proposition, we can assume that P and Q are as in (1.1). Let C be as in (1.2). Thus $\deg(C) = 1$, $C^m = P$ and C has the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \dots \quad \text{with each } C_{-i} \in K[y].$$

Since $m + n > 2$, by Proposition 1.7 there exist $j \in \mathbb{Z}$ and $\lambda \in K^\times$ such that

$$\deg(Q - \lambda C^j) < \deg(Q).$$

By (1.1) and (1.2), we have $j = m$ and $\lambda = 1$. We claim that there exist $\lambda_1, \dots, \lambda_{m+n-3} \in K$ such that

$$\deg(Q - C^m - \lambda_1 C^{m-1} - \dots - \lambda_{m+n-3} C^{3-n}) \leq 2 - n. \quad (1.3)$$

Assume he have found $\lambda_1, \dots, \lambda_i \in K$, where $i < m + n - 2$, such that

$$\deg(Q - C^m - \lambda_1 C^{m-1} - \dots - \lambda_i C^{m-i}) \leq m - i - 1 \quad (1.4)$$

Let $\tilde{Q} := Q - C^m - \lambda_1 C^{m-1} - \dots - \lambda_i C^{m-i}$. If

$$n + \deg(\tilde{Q}) - 2 = \deg(P) + \deg(\tilde{Q}) - |(1, 1)| \leq 0,$$

then we take $\lambda_{i+1} = \dots = \lambda_{m+n-3} = 0$. Otherwise,

$$\deg(\tilde{Q}) > 2 - n, \quad (1.5)$$

and, again by Proposition 1.7, there exist $j \in \mathbb{Z}$ and $\lambda_j \in K^\times$ such that

$$\deg(\tilde{Q} - \lambda_j C^{m-j}) < \deg(\tilde{Q}).$$

Consequently,

$$m - j = \deg(C^{m-j}) = \deg(\tilde{Q}),$$

and so, by (1.4) and (1.5),

$$i + 1 \leq j \leq m + n - 3.$$

This finishes the proof of the claim. Let

$$\tilde{F} := Q - C^m - \lambda_1 C^{m-1} - \dots - \lambda_{m+n-3} C^{3-n}.$$

Since $\deg(\tilde{F}) \leq 2 - n$, there exist $\tilde{F}_0, \tilde{F}_1, \dots \in K[y]$ with $\deg(\tilde{F}_i) \leq i$, such that

$$\tilde{F} = \tilde{F}_0 x^{2-n} + \tilde{F}_1 x^{1-n} + \tilde{F}_2 x^{-n} + \dots.$$

Setting $\lambda_{m+n-2} := \tilde{F}_0$ we obtain that

$$Q = C^m + \lambda_1 C^{m-1} + \dots + \lambda_{m+n-3} C^{3-n} + \lambda_{m+n-2} C^{2-n} + F, \quad (1.6)$$

where

$$F := \tilde{F} - \lambda_{m+n-2} C^{2-n} = F_1 x^{1-n} + F_2 x^{-n} + F_3 x^{-n-1} + \dots, \quad (1.7)$$

where $F_i \in K[y]$ and $\deg(F_i) \leq i$. Hence $\deg_x(F) \leq 1 - n$ and $F_1 = \mu_0 + \mu_1 y$ with $\mu_0, \mu_1 \in K$. Moreover since $P = C^m$ we have $[P, F] = [P, Q] \in K^\times$ and so, $\mu_1 \neq 0$, by Lemma 1.6. Let φ be the automorphism of $K[y]((x^{-1}))$ defined by

$$\varphi(x) := x \quad \text{and} \quad \varphi(y) := \frac{y - \mu_0}{\mu_1}.$$

Replacing P , Q , C and F by $\varphi(P)$, $\varphi(Q)$, $\varphi(C)$ and $\varphi(F)$, respectively, we can assume $\mu_0 = 0$ and $\mu_1 = 1$. Thus $F_+ = x^{1-n}y$, where F_+ is taken with respect to the $(1, 0)$ -grading. Note that this equality, combined with the fact that $\deg(F_i) \leq i$ for all i , gives $\deg(F) = 2 - n$.

\Leftarrow) Since

$$[P, F] - [P - P_+, F] - [P_+, F - F_+] = [P_+, F_+] = [x^n, x^{1-n}y] = n,$$

where P_+ and F_+ are taken with respect to the $(1, 0)$ -grading, and

$$\deg_x([P - P_+, F]), \deg_x([P_+, F - F_+]) < \deg_x(P) + \deg_x(F) - 1 \leq \deg(P) + \deg_x(F) - 1 = 0,$$

we have

$$[P, F] = n + \text{terms with } \deg_x \text{ lesser than } 0.$$

Moreover, using that

$$C^m = P \quad \text{and} \quad Q = \sum_{i=0}^{m+n-2} \lambda_i C^{m-i} + F,$$

we obtain that $\deg(P) = n$, $\deg(Q) = m$ and $[P, Q] = [P, F]$. Hence, neither $\deg(P)$ divides $\deg(Q)$ nor $\deg(Q)$ divides $\deg(P)$ and, since $[P, Q] \in R$, we also have

$$[P, Q] = [P, F] = n \in K^\times.$$

Consequently, by Proposition 1.1 the JC is false. \square

Remark 1.10. The proof of the theorem shows that if (P, Q) is a Jacobian pair such that neither $\deg(P)$ divides $\deg(Q)$ nor $\deg(Q)$ divides $\deg(P)$, then there is an affine change of variables that transforms it into a pair that satisfies the conditions of the statement of Theorem 1.9. Note that a such change of variables does not change neither $\deg(P)$ nor $\deg(Q)$.

Definition 1.11. Let D be a K -algebra, $n, m \in \mathbb{N}$ such that $n \nmid m$ and $m \nmid n$, $(\lambda_i)_{1 \leq i \leq n+m-2}$ a family of elements of K with $\lambda_0 = 1$ and $F_{1-n} \in D$. We say that $C \in D((x^{-1}))$ is a *solution* of the system $S(n, m, (\lambda_i), F_{1-n})$, if C has the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \quad \text{with each } C_{-i} \in D,$$

and there exist $P, Q \in D[x]$ and $F \in D[[x^{-1}]]$, such that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + F_{-1-n}x^{-1-n} + \cdots, \quad (1.8)$$

$$P = C^m \quad \text{and} \quad Q = \sum_{i=0}^{m+n-2} \lambda_i C^{m-i} + F. \quad (1.9)$$

Note that the polynomial Q does not depend on F since $\deg_x(F) < 0$. We say that (P, Q) is the *pair associated with the solution* C and we call P, Q the *polynomials associated with the solution* C .

From now on, when we mention a system $S(n, m, (\lambda_i), F_{1-n})$, unless otherwise specified, we will assume that $n \nmid m$ and $m \nmid n$.

Corollary 1.12. *The Jacobian conjecture is false if and only if for $D := K[y]$ there exist*

- $n, m \in \mathbb{N}$, such that $n \nmid m$ and $m \nmid n$,
- a family $(\lambda)_{0 \leq i \leq m+n}$ of elements of K with $\lambda_0 = 1$,
- a solution $C \in D((x^{-1}))$ of $S(n, m, (\lambda_i), y)$ such that

$$\deg(C) = 1 \quad \text{and} \quad \deg(F) = 2 - n,$$

where F is as in Definition 1.11.

Let A be an arbitrary K -algebra. In the sequel for each $E \in A((x^{-1}))$ and $k \in \mathbb{Z}$ we let E_k denote the coefficient of x^k in E .

Remark 1.13. Let $S(n, m, (\lambda_i), F_{1-n})$ be as in Definition 1.11 and let A be the polynomial K -algebra $D[Z_{-1}, Z_{-2}, Z_{-3}, \dots]$ in the indeterminates Z_v , with $v < 0$. Consider the Laurent series

$$Z := x + Z_{-1}x^{-1} + Z_{-2}x^{-2} + \dots \in A((x^{-1})).$$

If $C \in D((x^{-1}))$ is a solution of $S(n, m, (\lambda_i), F_{1-n})$, then the coefficients $C_{-1}, \dots, C_{-m-n+2}$ satisfy the $m+n-2$ equations

$$\begin{aligned} (Z^n)_{-k} &= 0, \quad \text{for } k = 1, \dots, m-1, \\ \left(\sum_{i=0}^{m+n-2} \lambda_i Z^{m-i} \right)_{-k} &= 0, \quad \text{for } k = 1, \dots, n-2, \\ \left(\sum_{i=0}^{m+n-2} \lambda_i Z^{m-i} \right)_{1-n} &+ F_{1-n} = 0. \end{aligned} \tag{1.10}$$

(Note that Z_{-n-m+2} is the lowest degree coefficient of Z which appears in the system. It appears in the equation $(Z^n)_{1-m} = 0$ and in the last equation).

Conversely, if $C_{-1}, \dots, C_{-m-n+2} \in D$ satisfy the equation system (1.10), then there exist unique

$$C_{-m-n+1}, C_{-m-n}, C_{-m-n-1}, \dots \in D,$$

such that

$$C := x + C_{-1}x^{-1} + C_{-2}x^{-2} + C_{-3}x^{-3} + \dots \tag{1.11}$$

is a solution of $S(n, m, (\lambda_i), F_{1-n})$. In fact, let $j \in \mathbb{N}_0$ and assume we have proven that there exist unique

$$C_{-m-n-i+2} \in D \quad \text{where } i \text{ runs from } 1 \text{ to } j,$$

such that $C_{-1}, \dots, C_{-m-n-j+2}$ satisfy

$$(Z^n)_{-k} = 0 \quad \text{for } k = 1, \dots, m-1+j. \tag{1.12}$$

Since

$$(Z^n)_{-m-j} = H + nZ_{-m-n-j+1},$$

where H is a sum of monomials of $K[Z_{-1}, \dots, Z_{-m-n-j+2}]$, we can solve $Z_{-m-n-j+1}$ univocally in the equation

$$0 = H(C_{-1}, \dots, C_{-m-n-j+2}) + nZ_{-m-n-j+1}.$$

So, there exists a unique $C_{-m-n-j+1} \in D$ such that $C_{-1}, \dots, C_{-m-n-j+1}$ satisfy

$$(Z^n)_{-m-j} = 0.$$

It is evident that $(C_{-1}, \dots, C_{-m-n-j})$ satisfies the system of equations (1.12), since $Z_{-m-n-j+1}$ does not appear in that system. In order to finish the proof we only must note that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + F_{-1-n}x^{-1-n} + \dots,$$

is univocally determined by the equations

$$\left(\sum_{i=0}^{m+n-2} \lambda_i Z^{m-i} \right)_{-k} + F_{-k} = 0 \quad \text{for } k \geq n.$$

Definition 1.14. We will write $S_t(n, m, (\lambda_i), F_{1-n})$ to denote the system of equations (1.10), and we call it the *(standard) system of equations associated with $S(n, m, (\lambda_i), F_{1-n})$* .

Definition 1.15. Given a solution $C_{-1}, \dots, C_{-m-n+2} \in D$ of (1.10), we call (1.11) the *solution of $S(n, m, (\lambda_i), F_{1-n})$ determined by $C_{-1}, \dots, C_{-m-n+2}$* .

Remark 1.16. Assume that $D = K[y]$. Let $S(n, m, (\lambda_i), y)$ be as in Corollary 1.12 and let

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + C_{-3}x^{-3} + \cdots \in D((x^{-1}))$$

be a solution of $S(n, m, (\lambda_i), y)$. Note that for $j > -m$,

$$\begin{aligned} 0 &= (C^n)_{-m-j} \\ &= \sum_{i_1 + \cdots + i_n = -m-j} C_{i_1} C_{i_2} \cdots C_{i_n} \\ &= n C_{-m-n-j+1} + \sum_{\substack{i_1 + \cdots + i_n = -m-j \\ i_k \neq -m-n-j+1 \ \forall k}} C_{i_1} C_{i_2} \cdots C_{i_n}, \end{aligned}$$

where we set $C_1 = 1$. From this it follows by induction that if

$$\deg(C_{-k}) \leq k + 1 \quad \text{for } k = 1, 2, \dots, m + n - 2, \quad (1.13)$$

then

$$\deg(C_{-k}) \leq k + 1 \quad \text{for all } k \geq 1. \quad (1.14)$$

Note also that equality in (1.13) implies equality in (1.14). A similar argument proves that under the same hypothesis,

$$\deg(F_{-k}) \leq 2 - n + k \quad \text{for all } k \geq n.$$

Resuming the results of this section we have the following corollary.

Corollary 1.17. *There exists a counterexample (P, Q) to JC with $(\deg(P), \deg(Q)) = (m, n)$ if and only if there exist $\lambda_1, \dots, \lambda_{m+n-2} \in K$ such that the standard system $S_t(n, m, (\lambda_i), y)$ has a solution in $K[y]^{m+n-2}$.*

2 Properties of solutions of $S(n, m, (\lambda_i), F_{1-n})$

In this section we show that under suitable conditions the system $S(n, m, (\lambda_i), F_{1-n})$ has only finitely many solutions. This applies in particular to the case related with the Jacobian conjecture.

Lemma 2.1. *Let Z be as in Remark 1.13. For all $i \in \mathbb{N}$ and $k, l \in \mathbb{Z}$, the equality*

$$\frac{\partial(Z^i)_k}{\partial Z_l} = i(Z^{i-1})_{k-l}$$

holds.

Proof. Since

$$\sum_k \frac{\partial(Z^i)_k}{\partial Z_l} x^k = \frac{\partial(\sum_k (Z^i)_k x^k)}{\partial Z_l} = \frac{\partial Z^i}{\partial Z_l} = i Z^{i-1} \frac{\partial Z}{\partial Z_l} = i Z^{i-1} x^l = \sum_j i(Z^{i-1})_j x^{j+l},$$

we have

$$\frac{\partial(Z^i)_k}{\partial Z_l} = i(Z^{i-1})_{k-l},$$

as desired. □

Let $S(n, m, (\lambda_i), F_{1-n})$ be as in Definition 1.11. Let Z and A be as in Remark 1.13. Consider the polynomials

$$E_1, \dots, E_{m+n-2} \in D[Z_{-1}, \dots, Z_{-m-n+2}],$$

defined by

$$E_i := \begin{cases} (Z^n)_{-i} & \text{for } 1 \leq i < m, \\ \left(\sum_{k=0}^{m+n-2} \lambda_k Z^{m-k} \right)_{m-i-1} & \text{for } m \leq i < m+n-2, \\ \left(\sum_{k=0}^{m+n-2} \lambda_k Z^{m-k} \right)_{1-n} + F_{1-n} & \text{for } i = m+n-2, \end{cases}$$

and set

$$J := \begin{pmatrix} \frac{\partial E_1}{\partial Z_{-1}} & \cdots & \frac{\partial E_1}{\partial Z_{-m-n+2}} \\ \vdots & \ddots & \vdots \\ \frac{\partial E_{m+n-2}}{\partial Z_{-1}} & \cdots & \frac{\partial E_{m+n-2}}{\partial Z_{-m-n+2}} \end{pmatrix}.$$

Note that since J is a matrix in $D[Z_{-1}, \dots, Z_{-m-n+2}]$ it makes sense to evaluate it in the tuple $(C_{-1}, \dots, C_{-m-n+2})$. Let

$$G := \sum_{k=0}^{m+n-2} \lambda_k (m-k) Z^{m-k-1}. \quad (2.1)$$

By the previous lemma we know that

$$\frac{\partial E_i}{\partial Z_{-j}} = \begin{cases} n(Z^{n-1})_{j-i} & \text{for } 1 \leq i < m, \\ G_{m+j-i-1} & \text{for } m \leq i < m+n-1. \end{cases}$$

Since

$$\deg(Z^{n-1}) = n-1 \quad \text{and} \quad \deg(G) = m-1,$$

this implies that J is the matrix $(Y_{ij}) \in M_{m+n-2}(A)$ given by

$$Y_{ij} := \begin{cases} n(Z^{n-1})_{j-i} & \text{if } 1 \leq i < m \text{ and } 1 \leq j < n+i, \\ G_{m+j-i-1} & \text{if } m \leq i < m+n-1 \text{ and } 1 \leq j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

In other words

$$J = \begin{pmatrix} J^{(1)} \\ J^{(2)} \end{pmatrix}, \quad (2.2)$$

where

$$J^{(1)} \in M_{(m-1) \times (m+n-2)}(A) \quad \text{and} \quad J^{(2)} \in M_{(n-1) \times (m+n-2)}(A) \quad (2.3)$$

are the matrices

$$J^{(1)} := \begin{pmatrix} n(Z^{n-1})_0 & \cdots & n(Z^{n-1})_{n-1} & 0 \\ \vdots & \ddots & \vdots & \ddots \\ n(Z^{n-1})_{2-m} & \cdots & n(Z^{n-1})_{n-m+1} & \cdots & n(Z^{n-1})_{n-1} \end{pmatrix}$$

and

$$J^{(2)} := \begin{pmatrix} G_0 & \cdots & G_{m-1} & 0 \\ \vdots & \ddots & \vdots & \ddots \\ G_{2-n} & \cdots & G_{m-n+1} & \cdots & G_{m-1} \end{pmatrix},$$

respectively.

For each $M \in M_{r \times s}(D)$, we let \overline{M} denote the k -linear map, from D^s to D^r , given by

$$\overline{M}(V) := (MV^t)^t \in D^r,$$

where, as usual, X^t denotes the transpose of X . In order to prove Theorem 2.3 below, we need to introduce some auxiliary maps.

Definition 2.2. We define the maps

$$\begin{aligned} \Pi_1: D((x^{-1})) &\rightarrow D^{m-1} & \text{by} & \quad \Pi_1(f) := (f_{-1}, f_{-2}, \dots, f_{1-m}), \\ \Pi_2: D((x^{-1})) &\rightarrow D^{n-1} & \text{by} & \quad \Pi_2(f) := (f_{-1}, f_{-2}, \dots, f_{1-n}), \\ \Gamma_1: D^{m-1} &\rightarrow D((x^{-1})) & \text{by} & \quad \Gamma_1(d_1, \dots, d_{1-m}) := d_1 x^{-1} + \dots + d_{1-m} x^{1-m}, \\ \Gamma_2: D^{n-1} &\rightarrow D((x^{-1})) & \text{by} & \quad \Gamma_2(d_1, \dots, d_{1-n}) := d_1 x^{-1} + \dots + d_{1-n} x^{1-n}. \end{aligned}$$

Note that Γ_1 and Γ_2 are right inverses to Π_1 and Π_2 , respectively. We will also need the map

$$\Pi: D((x^{-1})) \longrightarrow D^{m+n-2},$$

defined by $\Pi(f) := (f_{-1}, f_{-2}, \dots, f_{-m-n+2})$, and the canonical projections

$$\Pi_+: D((x^{-1})) \rightarrow D[x] \quad \text{and} \quad \Pi_-: D((x^{-1})) \rightarrow D[[x^{-1}]].$$

Theorem 2.3. Let $S(n, m, (\lambda_i), F_{1-n})$ be as in Definition 1.11,

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \dots \in D((x^{-1}))$$

a solution of $S(n, m, (\lambda_i), F_{1-n})$ and (P, Q) the pair associated with C . Assume that D is an integral domain over K . If there exist $A, B \in D[x]$ such that $AP' + BQ' = 1$, then the matrix $J|_{\mathfrak{v}}$, obtained evaluating J in

$$\mathfrak{v} := (C_{-1}, \dots, C_{-m-n+2}) \in D^{m+n-2},$$

is invertible.

Proof. Recall that $P = C^n$ and that there exists $F \in D((x^{-1}))$ such that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + F_{-1-n}x^{-1-n} + \dots, \quad (2.4)$$

and

$$Q = \sum_{i=0}^{m+n-2} \lambda_i C^{m-i} + F.$$

Let G be as in (2.1). Note that

$$G(C) = \sum_{i=0}^{m+n-2} \lambda_i (m-i) C^{m-i-1}$$

satisfies

$$G(C)C' = Q' - F'. \quad (2.5)$$

We claim that if $V \in D^{m+n-2}$ and $U \in D((x^{-1}))$ satisfy $\deg_x(U) < 0$ and $\Pi(U) = V$, then

$$\overline{J_{|_{\mathfrak{v}}}^{(1)}}(V) = \Pi_1(nC^{n-1}U) \quad \text{and} \quad \overline{J_{|_{\mathfrak{v}}}^{(2)}}(V) = \Pi_2(G(C)U). \quad (2.6)$$

Let $\overline{J_{|_{\mathfrak{v}}}^{(1)}}(V)_i$ be the i -th coordinate of $\overline{J_{|_{\mathfrak{v}}}^{(1)}}(V)$. Write $V = (v_1, \dots, v_{m+n-2})$. Since

$$\deg_x(C^{n-1}) = n-1 \quad \text{and} \quad \deg_x(U) < 0,$$

we have

$$\begin{aligned} \Pi_1(nC^{n-1}U)_i &= \sum_{j \in \mathbb{Z}} n(C^{n-1})_j U_{-j-i} \\ &= n(C^{n-1})_{1-i} U_{-1} + \dots + n(C^{n-1})_{n-1} U_{1-n-i} \\ &= n(C^{n-1})_{1-i} v_1 + \dots + n(C^{n-1})_{n-1} v_{n+i-1} \\ &= \overline{J_{|_{\mathfrak{v}}}^{(1)}}(V)_i, \end{aligned}$$

proving the first equality in the claim. The second one is similar.

Now, given $v_1 \in D^{m-1}$ and $v_2 \in D^{n-1}$, we set

$$V := (v_1, v_2) \in D^{m+n-2}, \quad V_1 := \Gamma_1(v_1) \quad \text{and} \quad V_2 := \Gamma_2(v_2).$$

We are going to prove that $V \in \overline{J_v}(D^{m+n-2})$. Define $h \in D[x]$ by

$$h := \Pi_+(V_2 P' - V_1 Q'). \quad (2.7)$$

Note that

$$\deg_x(h) \leq \min(n-2, m-2) < m+n-2. \quad (2.8)$$

From $AP' + BQ' = 1$ we obtain $AhP' + BhQ' = h$. Since the leading term of P' is invertible, there exist unique $T, A_1 \in D[x]$ with

$$\deg_x(A_1) < \deg_x(P') = n-1, \quad (2.9)$$

such that $hB = TP' + A_1$. Let $A_2 := -Ah - TQ'$. A direct computation shows that

$$A_1 Q' - A_2 P' = h. \quad (2.10)$$

Using this equality, conditions (2.8) and (2.9), and that $\deg_x(Q') = m-1$, we obtain that

$$\deg_x(A_2) < m-1. \quad (2.11)$$

Note that nC^{n-1} and $G(C)$ have invertible leading terms and hence are invertible in $D((x^{-1}))$. Moreover, by the definition of Γ_1 and (2.9), we have

$$\deg_x(A_1 + V_1) < n-1, \quad (2.12)$$

which implies

$$\deg_x \left(\frac{A_1 + V_1}{nC^{n-1}} \right) \leq (n-2) - (n-1) = -1. \quad (2.13)$$

A similar computation gives

$$\deg_x \left(\frac{A_2 + V_2}{G(C)} \right) \leq -1. \quad (2.14)$$

On the other hand, by (2.5),

$$(A_1 + V_1)G(C)C' = (A_1 + V_1)(Q' - F') = A_1 Q' + V_1 Q' - A_1 F' - V_1 F',$$

and, by the fact that $P = C^n$ and equality (2.10),

$$(A_2 + V_2)nC^{n-1}C' = (A_2 + V_2)P' = A_1 Q' - h + V_2 P'.$$

So,

$$\begin{aligned} \Pi \left(\frac{A_2 + V_2}{G(C)} - \frac{A_1 + V_1}{nC^{n-1}} \right) &= \Pi \left(\frac{V_2 P' - V_1 Q' - h + (A_1 + V_1)F'}{nC^{n-1}G(C)C'} \right) \\ &= \Pi \left(\frac{\Pi_-(V_2 P' - V_1 Q') + (A_1 + V_1)F'}{nC^{n-1}G(C)C'} \right) \\ &= 0, \end{aligned}$$

where the second equality follows from (2.7) and the last one from the facts that

$$\deg_x(nC^{n-1}G(C)C') = (n-1) + (m-1) = m+n-2.$$

and, by (2.4) and (2.12),

$$\deg_x((A_1 + V_1)F') \leq -1.$$

We set

$$X := \Pi \left(\frac{A_2 + V_2}{G(C)} \right) = \Pi \left(\frac{A_1 + V_1}{nC^{n-1}} \right).$$

Then, by (2.13), (2.14), (2.6) and the fact that A_1 and A_2 are polynomials, we have

$$\overline{J_{|v}^{(1)}}(X) = \Pi_1 \left(\frac{A_1 + V_1}{nC^{n-1}} nC^{n-1} \right) = \Pi_1(A_1 + V_1) = \Pi_1(V_1) = v_1$$

and

$$\overline{J_{|v}^{(2)}}(X) = \Pi_2 \left(\frac{A_2 + V_2}{G(C)} G(C) \right) = \Pi_2(A_2 + V_2) = \Pi_2(V_2) = v_2,$$

which finishes the proof. \square

Corollary 2.4. *Assume that we are under the hypothesis of Theorem 2.3 and that D is an integral domain. Then $S(n, m, (\lambda_i), F_{1-n})$ has finitely many solutions.*

Proof. Let L be an algebraic closure of the field of fractions of D . By the Jacobian Criterion, applying Theorem 2.3 with D replaced by L , we obtain that the set of solutions

$$(C_{-1}, \dots, C_{-m-n+2}) \in L^{m+n-2}$$

of $S(n, m, (\lambda_i), F_{1-n})$ is a zero-dimensional algebraic variety, and hence finite. \square

Remark 2.5. If $F_{1-n} = y$ and (P, Q) is a counterexample to JC, then the hypothesis of Theorem 2.3 are fulfilled with $A := \frac{\partial Q}{\partial y}$ and $B := -\frac{\partial P}{\partial y}$.

3 The homogeneous system $S(n, m, F_{1-n})$

In this section we let $S(n, m, F_{1-n})$ denote the system $S(n, m, (\lambda_i), F_{1-n})$ with $\lambda_i = 0$ for all $i \neq 0$, and we begin the study of the solution set of this system. Consider the polynomials

$$E_1^{(h)}, \dots, E_{m+n-2}^{(h)} \in K[Y][Z_{-1}, \dots, Z_{-m-n+2}],$$

defined by

$$E_i^{(h)} := \begin{cases} (\overline{Z}^n)_{-i} & \text{for } 1 \leq i < m, \\ (\overline{Z}^m)_{m-i-1} & \text{for } m \leq i < m+n-2, \\ (\overline{Z}^m)_{1-n} + Y^{m+n-1} & \text{for } i = m+n-2, \end{cases} \quad (3.1)$$

where

$$\overline{Z} := x + Z_{-1}x^{-1} + \dots + Z_{-m-n+2}x^{-m-n+2} \in K[Y][Z_{-1}, \dots, Z_{-m-n+2}]((x^{-1})).$$

Let $I^{(h)}$ be the ideal of $K[Y, Y^{-1}][Z_{-1}, \dots, Z_{-m-n+2}]$ generated by the $E_i^{(h)}$'s. Consider the weight w on the variables given by $w(Z_{-k}) := k+1$ and $w(Y) := 1$ and let wdeg denote the corresponding degree. Similar computations as in Remark 1.16 show that each $E_i^{(h)}$ is w -homogeneous with

$$\text{wdeg}(E_i^{(h)}) = \begin{cases} i+n & \text{if } i < m, \\ i+1 & \text{if } i \geq m. \end{cases}$$

Propositions 4.1 and 4.3 show that the system $S(n, m, Y^{m+n-1})$ has always a solution. In the present section we do not need this result.

Lemma 3.1. *If there exists a solution*

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + C_{-3}x^{-3} + \dots \in K[Y]((x^{-1}))$$

of the system $S(n, m, Y^{m+n-1})$, then for $k=1, \dots, m+n-2$ there exist $s_k \in \mathbb{N}$ and a w -homogeneous polynomial $h_k \in K[Y, Z_{-k}] \cap I^{(h)}$ with leading term $(Z_{-k})^{s_k}$, with respect to the graduation obtained giving weight 1 to Z_{-k} and 0 to Y .

Proof. Throughout this proof we write Z_{-k}^u instead of $(Z_{-k})^u$, and we let $[R, S]$ denote the Jacobian $J_{x,Y}(R, S)$ with respect to the variables x and Y . Let P and Q be the polynomials associated with C and let F be as in Definition 1.11. Let P_+ and F_+ be the leading terms of P and F with respect to \deg_x . Since, by (1.8) and (1.9),

$$[P_+, F_+] = [x^n, x^{1-n}Y^{m+n-1}] = n(m+n-1)Y^{m+n-2}, \quad (3.2)$$

we have

$$\begin{aligned} \deg_x([P - P_+, F]) &\leq \deg_x(P - P_+) + \deg_x(F) - 1 \\ &< \deg_x(P) + \deg_x(F) - 1 \\ &= n + (1 - n) - 1 \\ &= \deg_x([P_+, F_+]), \end{aligned} \quad (3.3)$$

and similarly,

$$\deg_x([P_+, F - F_+]) < \deg_x([P_+, F_+]). \quad (3.4)$$

Using (3.2), (3.3), (3.4) and that

$$[P_+, F_+] + [P - P_+, F] + [P_+, F - F_+] = [P, F],$$

we obtain

$$[P, Q] = [P, F] = [P_+, F_+] = [x^n, x^{1-n}Y^{m+n-1}] = n(m+n-1)Y^{m+n-2}, \quad (3.5)$$

where the first equality follows from (1.9). Let D be an algebraic closure of $K(Y)$. By (3.5), if we set

$$A := \frac{1}{n(m+n-1)Y^{m+n-2}} \frac{\partial Q}{\partial Y} \quad \text{and} \quad B := \frac{-1}{n(m+n-1)Y^{m+n-2}} \frac{\partial P}{\partial Y},$$

then

$$AP' + BQ' = 1.$$

By theorem 2.3 the set of all the solutions of the system of equations $S_t(n, m, Y^{m+n-1})$, introduced in Definition 1.14, is finite. For each $k \in \{1, \dots, m+n-2\}$, let

$$f := \prod_{j=1}^r (Z_{-k} - a_j) \in D[Z_{-k}] \subseteq D[Z_{-1}, \dots, Z_{-m-n+2}],$$

where $\{a_1, \dots, a_r\} \subseteq D$ is the set formed by the k th coordinates of the solutions in D^{m+n-2} of the system of equations mentioned above. Let $\bar{T}^{(h)}$ be the extension of $I^{(h)}$ in $D[Z_{-1}, \dots, Z_{-m-n+2}]$. By the nullstellensatz $f \in \sqrt{\bar{T}^{(h)}}$, and so, there is $t \in \mathbb{N}$ such that $f^t \in \bar{T}^{(h)}$. This means that

$$f^t = \sum_i \hat{f}_i E_i^{(h)}, \quad \text{for some } \hat{f}_i \in D[Z_{-1}, \dots, Z_{-m-n+2}]. \quad (3.6)$$

Let K_1 be the finite extension of $K(Y)$ generated by a_1, \dots, a_r . By the definition of f there exist $b_0, \dots, b_{rt-1} \in K_1$ such that

$$f^t = Z_{-k}^{rt} + b_{rt-1} Z_{-k}^{rt-1} + \dots + b_1 Z_{-k} + b_0. \quad (3.7)$$

Let e_0, \dots, e_T be a basis of K_1 over $K(Y)$ with $e_0 = 1$. Write

$$f^t = \sum_{l=0}^T h^{(l)} e_l, \quad \hat{f}_i = \sum_{l=0}^T f_i^{(l)} e_l \quad \text{and} \quad b_j = \sum_{l=0}^T b_j^{(l)} e_l,$$

where

$$h^{(l)} \in K(Y)[Z_{-k}], \quad f_i^{(l)} \in K(Y)[Z_{-1}, \dots, Z_{-m-n+2}] \quad \text{and} \quad b_j^{(l)} \in K(Y).$$

Using (3.6), (3.7), that $\hat{f}_i = \sum_{l=0}^T f_i^{(l)} e_l$ and that $e_0 = 1$, we obtain

$$\sum_i f_i^{(0)} E_i^{(h)} = Z_{-k}^{rt} + b_{rt-1}^{(0)} Z_{-k}^{rt-1} + \cdots + b_1^{(0)} Z_{-k} + b_0^{(0)}. \quad (3.8)$$

Consider the canonical inclusion of $K(Y)$ into $K((Y^{-1}))$ and write

$$b_i^{(0)} = \sum_{j \in \mathbb{Z}} \lambda_{ij} Y^j \quad \text{and} \quad f_i^{(0)} = \sum_{j \in \mathbb{Z}, \mathbf{l} \in \mathbb{N}_0^{m+n-2}} \gamma_{j,\mathbf{l}} Y^j \mathbf{Z}^{\mathbf{l}},$$

where

$$\mathbf{Z}^{\mathbf{l}} := Z_{-1}^{l_1} \cdots Z_{-m-n+2}^{l_{m+n-2}} \quad \text{if } \mathbf{l} = (l_1, \dots, l_{m+n-2}).$$

Set

$$f_i := \begin{cases} \sum_{(j,\mathbf{l}) \in \mathcal{B}_{i+n}} \gamma_{j,\mathbf{l}} Y^j \mathbf{Z}^{\mathbf{l}} & \text{if } i < m, \\ \sum_{(j,\mathbf{l}) \in \mathcal{B}_{i+1}} \gamma_{j,\mathbf{l}} Y^j \mathbf{Z}^{\mathbf{l}} & \text{if } i \geq m, \end{cases}$$

where $\mathcal{B}_u := \{(j, \mathbf{l}) : \text{wdeg}(Y^j \mathbf{Z}^{\mathbf{l}}) = rt(k+1) - u\}$. Note that f_i is the w -homogeneous component of $f_i^{(0)}$ satisfying

$$\text{wdeg}(f_i) + \text{wdeg}(E_i^{(h)}) = rt(k+1) = \text{wdeg}(Z_{-k}^{rt})$$

Taking the w -homogeneous component of degree $rt(k+1)$ in equality (3.8), we obtain

$$\sum_i f_i E_i^{(h)} = Z_{-k}^{rt} + \sum_{j=1}^{rt} \lambda_{rt-j, jk+j} Y^{jk+j} Z_{-k}^{rt-j}.$$

and so $s_k := rt$ and $h_k := \sum_i f_i E_i^{(h)}$ satisfy the required conditions. \square

Theorem 3.2. *Assume that $C \in K[Y]((x^{-1}))$ is a solution of $S(n, m, Y^{m+n-1})$. Then, for each $k = 1, \dots, m+n-2$ there exists $c_{-k} \in K$ such that*

$$C_{-k} = c_{-k} Y^{k+1}.$$

Proof. Let $h_k(Z_{-k}) \in K[Y][Z_{-k}]$ and s_k be as in the previous lemma. Since h_k is w -homogeneous,

$$h_k(Z_{-k}) = \sum_{i=r}^{s_k} \mu_i Y^{(s_k-i)(k+1)} Z_{-k}^i \quad \text{with } \mu_r \neq 0 \text{ and } \mu_{s_k} = 1.$$

Since $h_k \in I^{(h)}$, we know that $h_k(C_{-k}) = 0$. Suppose $C_{-k} \neq 0$ and write

$$C_{-k} = \sum_{j=t}^u \nu_j Y^j \quad \text{with } \nu_t, \nu_u \in K^\times.$$

In order to finish the proof we must check that $u = t = k+1$. But if $k+1 < u$, then

$$h_k(C_{-k}) = \mu_{s_k} \nu_u^{s_k} Y^{us_k} + \text{lower order terms},$$

and consequently $h_k(C_{-k}) \neq 0$, a contradiction. Similarly, if $t < k+1$, then

$$h_k(C_{-k}) = \mu_{s_k} \nu_t^{s_k} Y^{ts_k} + \text{higher order terms},$$

and consequently again $h_k(C_{-k}) \neq 0$. \square

By Remark 1.13 from the solutions of $S_t(n, m, 1)$ we obtain solutions of $S(n, m, 1)$. In the next section we will see that solutions of $S(n, m, 1)$ determine solutions of $S(n, m, Y^{n+m-1})$.

4 Presentations of the solutions of $S(n, m, Y^{m+n-1})$

In this section we focus on solutions of the system $S(n, m, Y^{m+n-1})$. This system has many different presentations. Note that if (P, Q) is the pair associated with a solution of $S(n, m, Y^{m+n-1})$, then by Theorem 3.2 and Remark 1.16,

$$P = C^m = \sum_{i=0}^n p_i x^i Y^{n-i} \quad \text{and} \quad Q = \Pi_+(C^m) = \sum_{i=0}^m q_i x^i Y^{m-i}$$

are homogeneous polynomials, with $p_n = q_m = 1$ and $p_{n-1} = q_{m-1} = 0$. Furthermore, by (3.5) we know that

$$[P, Q] = n(m+n-1)Y^{m+n-2}.$$

Proposition 4.1. *Let*

$$P = \sum_{i=0}^n p_i x^i Y^{n-i} \quad \text{and} \quad Q = \sum_{i=0}^m q_i x^i Y^{m-i}$$

be homogeneous polynomials with $p_n = q_m = 1$ and $p_{n-1} = 0$. Define $p, q \in K[x]$ by

$$p := \sum_{i=0}^n p_i x^i \quad \text{and} \quad q := \sum_{i=0}^m q_i x^i.$$

Let $\lambda \in K^\times$ and set $\tilde{\lambda} := n\lambda(1-m-n)$. The following conditions are equivalent:

- (1) *(P, Q) is the pair associated with a solution*

$$C := x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \in K[Y]((x^{-1}))$$

of the system $S(n, m, \lambda Y^{m+n-1})$.

- (2) *(p, q) is the pair associated with a solution*

$$c := x + c_{-1}x^{-1} + c_{-2}x^{-2} + \cdots \in K((x^{-1}))$$

of the system $S(n, m, \lambda)$.

- (3) $[P, Q] = \tilde{\lambda}Y^{m+n-2}$.

- (4) *The polynomials $p(x)$ and $q(x)$ fulfill*

$$mp'q - npq' = \tilde{\lambda}. \tag{4.9}$$

- (5) *The polynomials $p(x)$ and $q(x)$ fulfill*

$$p^m - q^n = n\lambda x^{mn-m-n+1} + \text{lower order terms}, \tag{4.10}$$

- (6) *Write*

$$p(x) = \prod_{i=1}^n (x - \alpha_i) \quad \text{and} \quad q(x) = \prod_{j=1}^m (x - \beta_j).$$

The polynomial $g := pq \in K[x]$ is separable and fulfills

$$mg'(\alpha_i) = \tilde{\lambda} \quad \text{and} \quad ng'(\beta_j) = -\tilde{\lambda}. \tag{4.11}$$

Proof. (1) \Leftrightarrow (2). This follows directly using the evaluation map at $Y = 1$ in one direction and taking $C_{-k} := c_{-k}Y^{k+1}$ in the other direction.

(2) \Rightarrow (5). We know that $p = c^n$ and there exists

$$f = \lambda x^{1-n} + f_{-n}x^{-n} + f_{-n-1}x^{-n-1} + \cdots \in K((x^{-1}))$$

such that $c^m = q + f$. Hence

$$p^m = c^{mn} = (q + f)^n = q^n + nq^{n-1}f + \binom{n}{2}q^{n-2}f^2 + \cdots$$

Since

$$\deg(q^{n-k}f^k) = m(n-k) + k(1-n) \quad \text{and} \quad q^{n-1}f = \lambda x^{mn-m-n+1} + \text{lower order terms.}$$

this implies item (5).

(5) \Rightarrow (2). An standard computation shows that there exists a unique

$$c = x + c_0 + c_{-1}x^{-1} + c_{-2}x^{-2} + \cdots \in K((x^{-1})),$$

such that $c^n = p$. Write $f := c^m - q$. Since the leading terms of q and c^m coincide, $\deg(f) < m$. Furthermore

$$c^{nm} = q^n + nfq^{n-1} + r,$$

where $r \in K[x]$ has degree lower than $\deg(fq^{n-1})$. On the other hand, by hypothesis,

$$c^{nm} - q^n = p^m - q^n = n\lambda x^{mn-m-n+1} + \text{lower order terms},$$

and so,

$$nfq^{n-1} + r = n\lambda x^{mn-m-n+1} + \text{lower order terms.}$$

Since q is monic of degree m , this implies that $\deg(f) = 1 - n$ and the principal coefficient f_{1-n} of f is λ .

(5) \Rightarrow (4). Set $j := mn - m - n$ and write

$$t := p^m - q^n - n\lambda x^{j+1}. \tag{4.12}$$

By hypothesis $\deg(t) \leq j$. Computing the derivative in (4.12), we obtain

$$mp^{m-1}p' = nq^{n-1}q' + (j+1)n\lambda x^j + t'.$$

Multiplying this equality by q , and dividing the result by p^{m-1} , we get

$$mqp' = nq' \frac{q^n}{p^{m-1}} + (j+1)n\lambda x^j \frac{q}{p^{m-1}} + t' \frac{q}{p^{m-1}}.$$

But, by (4.12)

$$\frac{q^n}{p^{m-1}} = p - \frac{n\lambda x^{j+1}}{p^{m-1}} - \frac{t}{p^{m-1}},$$

and so

$$mqp' = npq' - \frac{n^2\lambda x^{j+1}q'}{p^{m-1}} + (j+1)n\lambda x^j \frac{q}{p^{m-1}} - \frac{ntq'}{p^{m-1}} + t' \frac{q}{p^{m-1}}.$$

Since p and q are polynomials,

$$\deg\left(\frac{n^2\lambda x^{j+1}q'}{p^{m-1}}\right) = 0 \text{ and its principal coefficient is } n^2m\lambda,$$

$$\deg\left((j+1)n\lambda x^j \frac{q}{p^{m-1}}\right) = 0 \text{ and its principal coefficient is } (j+1)n\lambda$$

and

$$\deg\left(\frac{ntq'}{p^{m-1}}\right), \deg\left(t'\frac{q}{p^{m-1}}\right) < 0,$$

we conclude that

$$mqp' = npq' + n\lambda(1 - m - n),$$

as desired.

(4) \Rightarrow (5). By hypothesis

$$\left(\frac{p^m}{q^n}\right)' = \frac{mp^{m-1}p'q^n - nq^{n-1}q'p^m}{q^{2n}} = \frac{(mp'q - nq'p)q^{n-1}p^{m-1}}{q^{2n}} = \tilde{\lambda}\frac{p^{m-1}}{q^{n+1}}.$$

Since

$$\deg\left(\tilde{\lambda}\frac{p^{m-1}}{q^{n+1}}\right) = -m - n \text{ and its principal coefficient is } \tilde{\lambda},$$

there exist $\kappa \in K$ and $r \in K((x^{-1}))$ such that $\deg(r) = 1 - m - n$, the principal coefficient of r is $\tilde{\lambda}/(1 - m - n)$ and

$$\frac{p^m}{q^n} = \kappa + r.$$

Moreover, since $\deg(p^m) = \deg(q^n)$ and p, q are monic, $\kappa = 1$. Hence,

$$p^m = q^n + \frac{\tilde{\lambda}}{1 - m - n} x^{mn-m-n+1} + \text{terms of lower order},$$

as desired.

(3) \Leftrightarrow (4). A direct computation shows that

$$\begin{aligned} [P, Q] &= P_x Q_Y - P_Y Q_x \\ &= \sum_{i=0}^n ip_i x^{i-1} Y^{n-i} \sum_{j=0}^m (m-j) q_j x^j Y^{m-j-1} \\ &\quad - \sum_{i=0}^n (n-i) p_i x^i Y^{n-i-1} \sum_{j=0}^m j q_j x^{j-1} Y^{m-j} \\ &= \sum_{i,j} p_i q_j (i(m-j) - (n-i)j) x^{i+j-1} Y^{m+n-i-j-1} \\ &= \sum_{i,j} p_i q_j (mi - nj) x^{i+j-1} Y^{m+n-i-j-1} \end{aligned}$$

and

$$\begin{aligned} mp'q - npq' &= m \sum_{i=0}^n ip_i x^{i-1} \sum_{j=0}^m q_j x^j - n \sum_{i=0}^n p_i x^i \sum_{j=0}^m j q_j x^{j-1} \\ &= \sum_{i,j} p_i q_j m i x^{i+j-1} - \sum_{i,j} p_i q_j n j x^{i+j-1} \\ &= \sum_{i,j} p_i q_j (mi - nj) x^{i+j-1}. \end{aligned}$$

So, it is clear that $[P, Q] = \tilde{\lambda} Y^{m+n-2}$ if and only if $mp'q - npq' = \tilde{\lambda}$.

(4) \Rightarrow (6). A direct computation shows that

$$mg' - (m+n)pq' = mp'q - npq' = (m+n)qp' - ng'. \quad (4.13)$$

Using item (4) and evaluating the first equality at the α_i 's and the second one at the β_j 's, we obtain (4.11). Since $\lambda \neq 0$, this implies that g has no multiple roots, and so g is separable.

(6) \Rightarrow (4). By equalities (4.13) and the hypothesis, we have

$$(mp'q - npq')(\alpha_i) = (mp'q - npq')(\beta_j) = \tilde{\lambda} \quad \text{for all } i, j.$$

Since $\deg(mp'q - npq') \leq n + m - 1$, this implies that $mp'q - npq' = \tilde{\lambda}$. \square

Proposition 4.2. *Let $n, m > 1$. If $n|m$ or $m|n$, then there is no solution to (4.9).*

Proof. Assume that $mp'q - npq' = \tilde{\lambda}$ and $m = nk$ with $k \in \mathbb{N}$. Set $\bar{q} := q - p^k$. Then

$$\bar{q}(x) = a_r x^r + \text{lower degree terms} \quad \text{for some } 0 \leq r < m.$$

On one hand

$$mp'\bar{q} - np\bar{q}' = \tilde{\lambda},$$

but, on the other hand the leading term of $mp'\bar{q} - np\bar{q}'$ is $na_r x^{n+r-1}(m-r)$. Hence $n+r-1 = 0$, which contradicts $n > 1$ and $r \geq 0$. \square

Proposition 4.3. *If $m \nmid n$ and $n \nmid m$, then the system $S(n, m, \lambda)$ has at least one solution.*

Proof. Set $\mu_1 = \mu_2 = \dots = \mu_n := m$ and $\nu_1 = \nu_2 = \dots = \nu_m := n$. Clearly

$$mn = \sum_{i=1}^m \mu_i = \sum_{j=1}^n \nu_j.$$

Moreover $\delta := \gcd(m, n) < m, n$, which implies that

$$\max \left\{ mn \frac{\delta - 1}{\delta}, mn - m - n + 1 \right\} = mn - m - n + 1.$$

Hence, by [9, Theorem 1, page 114] there exist polynomials F, G having μ_i , resp. ν_j as the sequences of multiplicities of their roots, satisfying

$$\deg(F - G) = mn - m - n + 1,$$

and it is evident that we can assume that F and G are monic. But then $F(x) = p(x)^m$, where $p(x)$ is the product of the linear factors of F and similarly $G(x) = q(x)^n$ with $q(x)$ monic. Then

$$p(x)^m - q(x)^n = F - G = n\mu x^{mn-m-n+1} + \text{lower order terms}$$

for some $\mu \in K^\times$. Using the automorphism of $K[x]$ given by $x \mapsto x - p_{n-1}/n$ we achieve $p_{n-1} = 0$. Hence, the condition (5) of Proposition 4.1 is satisfied, and by that proposition the pair (p, q) is associated to a solution of $S(n, m, \mu)$. Let $\alpha \in K^\times$ be such that $\alpha^{n+m-1} = \lambda/\mu$. Replacing p_i by $\alpha^{n-i}p_i$ and q_i by $\alpha^{m-i}q_i$ for all i , we obtain a solution of $S(n, m, \lambda)$, as desired. \square

By definitions two pairs (p, q) and (p_1, q_1) of monic polynomials in $K[x]$ are ∞ -equivalent if there are $a \in K^\times$ and $b \in K$ such that

$$p_1(x) = a^{-\deg(p)} p(ax + b) \quad \text{and} \quad q_1(x) = a^{-\deg(q)} q(ax + b).$$

Theorem 2.3 and Propositions 4.1 and 4.2 show that $S_t(n, m, Y^{m+n-1})$ has finitely many solutions. This yields an alternative proof of a result contained in Theorem 4 of [2], which says that the equation (4.9) has only finitely many solutions for fixed m, n , modulo ∞ -equivalence. In fact we have:

Proposition 4.4. *Assume that K is algebraically closed and let m, n be positive integers. Then there are only finitely many ∞ -equivalence classes of pairs of monic polynomials $p, q \in K[x]$ such that p has degree n , q has degree m , and $mp'q - npq'$ is equal to $\tilde{\lambda}$ for some $\tilde{\lambda} \in K^\times$.*

Proof. Let \mathcal{S} be the set of pairs (p, q) of monic polynomials in $K[x]$ of degree n and m , respectively, such that

$$mp'q - npq' = 1 \quad \text{and} \quad p = x^n + p_{n-2}x^{n-2} + \cdots + p_0.$$

By Theorem 2.3 and Propositions 4.1 and 4.2 we know that \mathcal{S} is a finite set. So in order to finish the proof it suffices to show that if (\tilde{p}, \tilde{q}) is a pair of monic polynomials in $K[x]$ of degree n and m respectively such that

$$m\tilde{p}'\tilde{q} - n\tilde{p}\tilde{q}' = \tilde{\lambda},$$

where $\tilde{\lambda} \in K^\times$, then (\tilde{p}, \tilde{q}) is ∞ -equivalent to a pair $(p, q) \in \mathcal{S}$. But for this it suffices to take

$$p(x) := a^{-n}\tilde{p}(ax - \tilde{p}_{n-1}/n) \quad \text{and} \quad q(x) := a^{-n}\tilde{q}(ax - \tilde{p}_{n-1}/n),$$

where \tilde{p}_{n-1} is the coefficient of x^{n-1} in \tilde{p} and $a \in K$ satisfies $a^{m+n-1} = \tilde{\lambda}$. \square

Moreover, we have additional information about the set \mathcal{S}_0 of solutions (p, q) of (4.9) satisfying that p and q are monic, $\deg(p) = n$, $\deg(q) = m$ and the coefficient of x^{n-1} in p is zero. Let $e := m + n - 1$ and assume that K has a primitive e -root of unit. The group $\mathbb{Z}/e\mathbb{Z}$ acts on \mathcal{S}_0 . In fact, if $(c_{-1}, \dots, c_{-k}, \dots, c_{-m-n+2})$ is a solution of $S(n, m, \lambda)$ in K^{m+n-2} , then $(c_{-1}u^{2i}, \dots, c_{-k}u^{(k+1)i}, \dots, c_{-m-n+2}u^{(m+n-1)i})$ is also a solution of $S(n, m, \lambda)$ in K^{m+n-2} , and so we can define

$$i \cdot (c_{-1}, \dots, c_{-k}, \dots, c_{-m-n+2}) := (c_{-1}u^{2i}, \dots, c_{-k}u^{(k+1)i}, \dots, c_{-m-n+2}u^{(m+n-1)i}).$$

One can also check that if $n = 2$ and $m = 2r + 1$, then there are exactly $r + 1$ solutions (all in the same orbit). It is not clear in which cases there are orbits with $m + n - 1$ elements. We pose the following questions:

- (1) Let d be a divisor of $m + n - 1$ and assume $\{m \pmod{d}, n \pmod{d}\} = \{0, 1\}$. Does there exist always an orbit of solutions of $S(n, m, Y^{m+n-1})$ with $\frac{m+n-1}{d}$ elements, such that $C_{-k} = 0$ for $k + 1 \not\equiv 0 \pmod{d}$?
- (2) Let ϕ be the Euler function. If $\phi(m + n - 1) > 2$, does there exist an orbit in the solution set of $S(n, m, Y^{m+n-1})$ with $m + n - 1$ elements?

In [2] the author also considers the equation

$$mp'q - npq' = \lambda p \tag{4.14}$$

where $\lambda \in K^\times$. This equation is strongly related with equation (4.9) by the following:

$$mp'q - npq' = \lambda \implies (m + n)p'Q - npQ' = \lambda p,$$

where $Q := pq$.

For the rest of the section we will prove the following proposition, which answers partially question (2) in a particular case:

Proposition 4.5. *Let d be a divisor of $m + n - 1$ and let $r := \gcd(m, n)$. Assume that $d > r$ and that $\{m \pmod{d}, n \pmod{d}\} = \{0, 1\}$. Then there exists always a solution C of $S(n, m, 1)$ such that $C_{-k} = 0$ for $k + 1 \not\equiv 0 \pmod{d}$.*

Let A_1 be the polynomial K -algebra $K[Z_{-r-d}, Z_{r-2d}, Z_{r-3d}, \dots]$ in the variables Z_{r-vd} , with $v > 0$. Consider the Laurent series

$$Z := x^r + Z_{r-d}x^{r-d} + Z_{r-2d}x^{r-2d} + \cdots \in A_1((x^{-1})).$$

Set $N := (m + n - 1)/d$ and assume, without loss of generality, that

$$m = 1 \pmod{d} \quad \text{and} \quad n = 0 \pmod{d}.$$

Let $\lambda \in K$ and let $\tilde{C} \in K((x^{-1}))$ be a solution of $S(n, m, \lambda)$ with $\tilde{C}_{-k} = 0$ for $k+1 \not\equiv 0 \pmod{d}$. If we define $C := \tilde{C}^r$, then the coefficients C_{r-d}, \dots, C_{r-Nd} of C satisfy the N equations

$$\begin{aligned} G_k &:= (Z^{n/r})_{-dk} = 0, & \text{for } k = 1, \dots, (m-1)/d, \\ G_{k+(m-1)/d} &:= (Z^{m/r})_{-dk+1} = 0, & \text{for } k = 1, \dots, n/d-1, \\ G_N &:= (Z^{m/r})_{-n+1} + \lambda = 0. \end{aligned} \quad (4.15)$$

(Note that Z_{r-Nd} is the lowest degree coefficient of Z which appears in the system. It appears in the equation $(Z^{n/r})_{m-1} = 0$ and in the last equation).

Lemma 4.6. *Let $d := \gcd(n, m)$ and $j \in \mathbb{N}_0$. Let $P \in x^j K[x]$ be a monic polynomial of degree n and let*

$$C = x + C_0 x^0 + C_{-1} x^{-1} + C_{-2} x^{-2} + \dots \in K((x^{-1}))$$

be such that $C^m = P$. If $(C^m)_{-k} = 0$ for $k = 1, \dots, n - \max(j, 1)$, then $C^d \in K[x]$.

Proof. Write $C^m = Q + F$ where $Q \in K[x]$ and $F \in x^{-1} K[[x^{-1}]]$. Since $P \in x^j K[x]$, we have

$$G := mP'Q - nQ'P \in \begin{cases} x^{j-1} K[x] & \text{if } j > 0, \\ K[x] & \text{if } j = 0. \end{cases} \quad (4.16)$$

We claim that $G = 0$. Since,

$$G = mnC^{m-1}C'(C^m - F) - nC^m(mC^{m-1}C' - F') = nF'C^m - mnFC^{m-1}C'$$

and, by hypothesis, $\deg(F) \leq \max(j, 1) - n - 1$, if $G \neq 0$, then $\deg(G) \leq \max(j, 1) - 2$, which is impossible by equality (4.16). Thus the claim follows. But then

$$\left(\frac{P^m}{Q^n}\right)' = \frac{mP^{m-1}P'Q^n - nQ^{n-1}Q'P^m}{Q^{2n}} = \frac{P^{m-1}Q^{n-1}}{Q^{2n}}(mP'Q - nQ'P) = 0,$$

which combined with the fact that P and Q are monic, implies that $Q^n = P^m$. Consequently there exists a monic polynomial R such that $P = R^{n/d}$, and so $C^d = R \in K[x]$, as desired. \square

Proposition 4.7. *Let I be the ideal of $K[Z_{r-d}, \dots, Z_{r-Nd}]$ generated by $G_1, \dots, G_{N-1}, G^{(0)}$, where $G^{(0)} := (Z^{m/r})_{1-n}$. Then $\sqrt{I} = \langle Z_{r-d}, \dots, Z_{r-Nd} \rangle$.*

Proof. By the Nullstellensatz it suffices to prove that $V(I) = \{(0, \dots, 0)\}$, where $V(I)$ denotes the Zero-locus of the ideal I . So take a solution

$$c := (C_{r-d}, \dots, C_{r-Nd}) \in K^N$$

of $G_1, \dots, G_{N-1}, G^{(0)}$, and set

$$C := x^r + C_{r-d}x^{r-d} + C_{r-2d}x^{r-2d} + \dots + C_{r-Nd}x^{r-Nd} \in x^r K[[x^{-d}]].$$

Clearly

$$(C^{m/r})_{-k} = 0 \quad \text{for } k = 1, \dots, m-1 \quad \text{and} \quad (C^{m/r})_{-k} = 0 \quad \text{for } k = 1, \dots, n-1. \quad (4.17)$$

Now, by a similar argument as in Remark 1.13, there exists

$$C_{r-Nd-d}, C_{r-Nd-2d}, C_{r-Nd-3d}, \dots \in K,$$

such that the

$$\overline{C} := x^r + \sum_{k=1}^{\infty} C_{r-kd}x^{r-kd} \in x^r K[[x^{-d}]]$$

still satisfies (4.17) and such that the monic r -root of C ,

$$\tilde{C} := x + \tilde{C}_{1-d}x^{1-d} + \tilde{C}_{1-2d}x^{1-2d} + \tilde{C}_{1-3d}x^{1-3d} + \dots \in xK[[x^{-d}]]$$

is a solution of $S(n, m, 0)$. Hence $P := \tilde{C}^n$ is a monic polynomial of degree n and we can apply Lemma 4.6 with $j = 0$. Hence $\overline{C} = \tilde{C}^r \in K[x]$ and so, $C_{r-d} = 0, \dots, C_{r-Nd} = 0$ because $d > r$. This means that $c = (0, \dots, 0)$, as desired. \square

Corollary 4.8. *Let I_1 be the ideal of $K[Z_{r-d}, \dots, Z_{r-Nd}]$ generated by G_1, \dots, G_{N-1} . Then $G^{(0)} \notin \sqrt{I_1}$.*

Proof. If we assume that $G^{(0)} \in \sqrt{I_1}$, then by Proposition 4.7 we have $\sqrt{I_1} = \langle Z_{r-d}, \dots, Z_{r-Nd} \rangle$, which is impossible since I_1 is generated by $N-1$ elements and the height of $\langle Z_{r-d}, \dots, Z_{r-Nd} \rangle$ is N . \square

Proof of Proposition 4.5. By Corollary 4.8 and the Nullstellensatz, there exists

$$C = (C_{r-d}, \dots, C_{r-Nd}) \in K^N$$

such that $G_i(C) = 0$ for $1 \leq i < N$, but $G^{(0)}(C) \neq 0$. Let

$$\tilde{C} := x + \tilde{C}_{1-d}x^{1-d} + \tilde{C}_{1-2d}x^{1-2d} + \tilde{C}_{1-3d}x^{1-3d} + \dots \in xK[[x^{-d}]]$$

be the monic r -root in $xK[[x^{-d}]]$ of the Laurent series \overline{C} determined by C as in Remark 1.13. Then

$$(\tilde{C}_{-1}, \dots, \tilde{C}_{-Nd+1})$$

is a solution of $S_t(n, m, \lambda)$, where $\lambda := -G^{(0)}(C)$. Let $\alpha \in K$ be such that $\alpha^N = 1/\lambda$ and set $\hat{C}_{1-id} := \alpha^i \tilde{C}_{1-id}$. It is clear that $\hat{C} := (\hat{C}_{-1}, \dots, \hat{C}_{-Nd+1})$ is a solution of $S_t(n, m, 1)$. As in Remark 1.13, this determines a solution \tilde{C} of $S(n, m, 1)$. It is easy to check that $\tilde{C}_{-k} = 0$ for $k+1 \not\equiv 0 \pmod{d}$, as desired. \square

5 A modified system and an example

In this section we modify the system (1.10) in order to verify one of the 4 exceptional cases found by Moh in [7]. The case $(m, n) = (48, 64)$ has been already be verified independently in [5] and [4]. We will verify the case $(m, n) = (50, 75)$. Doing this directly using (1.10) amounts to solving a system of 123 equations and 123 variables. Due to this we take an alternative strategy. The first part of this procedure is similar to the one used in [3, Section 8], and is inspired by [7]. We do not provide proofs for this first part, since it serves only to verify a known case and to show the usefulness of systems like (1.10). Let A_0 and γ be as in the discussion above [3, Proposition 6.2]. Assume there is a counterexample (P_0, Q_0) to the Jacobian conjecture with $\deg(P_0) = 50$ and $\deg(Q_0) = 75$. Then by [3, Remark 7.10], we know that $A_0 = (5, 20)$. Furthermore, using similar computations as in [3, Proposition 8.3], one can check that necessarily $\gamma = 3$ or $\gamma = 2$. Proceeding as in [3, Section 8] we obtain a pair $(P_1, Q_1) \in K[x, y]$, such that

$$[P_1, Q_1] = x^2, \quad \deg(P_1) = 10 \quad \text{and} \quad \deg(Q_1) = 15.$$

If $\gamma = 3$, then applying to (P_1, Q_1) first the automorphism $x \mapsto xy^3, y \mapsto y^{-2}$ of $K[x, y, y^{-1}]$, and then the automorphism $x \mapsto x - G, y \mapsto y$ for some suitable $G \in K[y, y^{-1}]$, we obtain a pair $(P, Q) \in K[x, y, y^{-1}]$ satisfying:

(a1) There exist $\lambda \in K, \mu \in K^\times$ and $C, F \in K[y, y^{-1}]((x^{-1}))$ such that

$$P = C^2 \quad \text{and} \quad Q = C^3 + \lambda C^{-1} + F,$$

(a2) $[P, Q] = \mu y^6(x - G)^2$, for some $G \in K[y, y^{-1}]$,

(a3) there exists $f_2, f_4, f_6, f_8 \in K$ such that

$$F = F_{-1}x^{-1} + F_{-2}x^{-2} + F_{-3}x^{-3} + \cdots,$$

with $F_{-1} := y^7$ and $F_{-2} := f_8y^8 + f_6y^6 + f_4y^4 + f_2y^2$,

(a4) $C = x^2 + C_0 + C_{-1}x^{-1} + \cdots$,

(a5) $\deg_y(C_{-k}) \leq k + 2$ for all $k \geq 0$,

(a6) $C_0 = c_{0,2}y^2 + c_{0,0} + c_{0,-2}y^{-2} + \cdots + c_{0,-10}y^{-10}$, with $c_{0,-10} \neq 0$.

On the other hand, if $\gamma = 2$, then applying to (P_1, Q_1) first the automorphism $x \mapsto xy^2$, $y \mapsto y^{-3}$ of $K[x, y, y^{-1}]$, and then the automorphism $x \mapsto x - G$, $y \mapsto y$ for some suitable $G \in K[y, y^{-1}]$, we obtain a pair $(P, Q) \in K[x, y, y^{-1}]$ satisfying:

(b1) There exist $\lambda \in K$, $\mu \in K^\times$ and $C, F \in K[y, y^{-1}]((x^{-1}))$ such that

$$P = C^2 \quad \text{and} \quad Q = C^3 + \lambda C^{-1} + F,$$

(b2) $[P, Q] = \mu y^2(x - G)^2$, where $G := g_{-2}y^{-2} + g_{-5}y^{-5}$, with $g_{-2}, g_{-5} \in K$,

(b3) there exist $f_2, f_{-1}, f_{-4}, f_{-7}, b_1, b_{-2} \in K$ such that

$$F = F_{-3}x^{-3} + F_{-4}x^{-4} + F_{-5}x^{-5} + \cdots,$$

with $F_{-3} := y^3$, $F_{-4} := b_1y + b_{-2}y^{-2}$ and $F_{-5} := f_2y^2 + f_{-1}y^{-1} + f_{-4}y^{-4} + f_{-7}y^{-7}$,

(b4) $C = x^3 + C_1x + C_0 + C_{-1}x^{-1} + \cdots$,

(b5) $C_{-1} = c_{-1,1}y + c_{-1,-2}y^{-2} + \cdots + c_{-1,-17}y^{-17} + c_{-1,-20}y^{-20}$, with $c_{-1,1} \neq 0$,

(b6) $C_1 = e_{-1}y^{-1} + e_{-4}y^{-4} + e_{-7}y^{-7} + e_{-10}y^{-10}$ and $e_{-10} \neq 0$ if $C_0 = 0$.

We first analyze the case $\gamma = 3$. Motivated by (a4), we consider the Laurent series

$$Z := x^2 + Z_0 + Z_{-1}x^{-1} + Z_{-2}x^{-2} + \cdots \in K[Z_0, Z_{-1}, Z_{-2}, \dots]((x^{-1})).$$

We set

$$\begin{aligned} E_k &:= (Z^2)_{-k}, \quad \text{for } k = 1, \dots, 5, \\ E_{5+k} &:= (Z^3 + \lambda Z^{-1})_{-k}, \quad \text{for } k = 1, \dots, 3. \end{aligned} \tag{5.18}$$

Explicitly, we have

$$\begin{aligned} E_1 &= 2Z_0Z_{-1} + 2Z_{-3}, \\ E_2 &= Z_{-1}^2 + 2Z_0Z_{-2} + 2Z_{-4}, \\ E_3 &= 2Z_{-1}Z_{-2} + 2Z_0Z_{-3} + 2Z_{-5}, \\ E_4 &= Z_{-2}^2 + 2Z_{-1}Z_{-3} + 2Z_0Z_{-4} + 2Z_{-6}, \\ E_5 &= 2Z_{-2}Z_{-3} + 2Z_{-1}Z_{-4} + 2Z_0Z_{-5} + 2Z_{-7}, \\ E_6 &= 3Z_0^2Z_{-1} + 6Z_{-1}Z_{-2} + 6Z_0Z_{-3} + 3Z_{-5}, \\ E_7 &= \lambda + 3Z_0Z_{-1}^2 + 3Z_0^2Z_{-2} + 3Z_{-2}^2 + 6Z_{-1}Z_{-3} + 6Z_0Z_{-4} + 3Z_{-6}, \\ E_8 &= Z_{-1}^3 + 6Z_0Z_{-1}Z_{-2} + 3Z_0^2Z_{-3} + 6Z_{-2}Z_{-3} + 6Z_{-1}Z_{-4} + 6Z_0Z_{-5} + 3Z_{-7}. \end{aligned}$$

Note that Z_{-7} is the lowest degree coefficient of Z which appears in the E_i 's. It appears in the term $2Z_{-7}$ of E_5 and in the term $3Z_{-7}$ of E_8 . If $C \in K[y, y^{-1}]((x^{-1}))$ fulfills (a1)–(a6), then the 8 coefficients $C_1, C_0, C_{-1}, \dots, C_{-7}$, of C , satisfy the equations

$$E_1 = \cdots = E_5 = 0, \quad E_6 = -F_{-1}, \quad E_7 = -F_{-2} \quad \text{and} \quad E_8 = -F_{-3}.$$

From $E_1 = 0$, $E_3 = 0$ and $E_6 = -F_{-1}$ we obtain $F_{-1} + 3C_{-1}C_{-2} = 0$. Setting

$$F_{-1} := -3C_{-1}C_{-2}$$

and eliminating in the set of equations

$$E_2 = \cdots = E_5 = 0, \quad E_6 = -F_{-1} \quad \text{and} \quad E_7 = -F_{-2},$$

the variables C_{-3} , C_{-4} , C_{-5} , C_{-6} and C_{-7} , we obtain

$$C_0(3C_0C_{-1}^2 - 3C_{-2}^2 - 2\lambda) = 2C_0F_{-2}.$$

But using that $y^7 + 3C_{-1}C_{-2} = 0$ and that by (a5) we have $\deg_y(C_{-1}) \leq 3$ and $\deg_y(C_{-2}) \leq 4$, we get $C_{-1} = ay^3$ and $C_{-2} = by^4$ for some $a, b \in K^\times$. Hence, either $C_0 = 0$ or

$$C_0 = \frac{3C_{-2}^2 + 2F_{-2} + 2\lambda}{3C_{-1}^2} = \frac{2\lambda}{3a^2y^6} + \frac{2f_2}{3a^2y^4} + \frac{2f_4}{3a^2y^2} + \frac{2f_6}{3a^2} + \frac{b^2y^2}{a^2} + \frac{2f_8y^2}{3a^2},$$

which contradicts that by (a6) we have $c_{0,-10} \neq 0$. This rules out the case $\gamma = 3$.

We now analyze the case $\gamma = 2$. Motivated by (b4) we consider the Laurent series

$$Z := x^3 + Z_1x + Z_0 + Z_{-1}x^{-1} + Z_{-2}x^{-2} + \cdots \in K[Z_1, Z_0, Z_{-1}, Z_{-2}, \dots]((x^{-1})).$$

We set

$$\begin{aligned} E_k &:= (Z^2)_{-k}, \quad \text{for } k = 1, \dots, 8, \\ E_{8+k} &:= (Z^3 + \lambda Z^{-1})_{-k}, \quad \text{for } k = 1, \dots, 5. \end{aligned} \tag{5.19}$$

Explicitly we have

$$\begin{aligned} E_1 &= 2Z_0Z_{-1} + 2Z_1Z_{-2} + 2Z_{-4}, \\ E_2 &= (Z_{-1})^2 + 2Z_0Z_{-2} + 2Z_1Z_{-3} + 2Z_{-5}, \\ E_3 &= 2Z_{-1}Z_{-2} + 2Z_0Z_{-3} + 2Z_1Z_{-4} + 2Z_{-6}, \\ E_4 &= (Z_{-2})^2 + 2Z_{-1}Z_{-3} + 2Z_0Z_{-4} + 2Z_1Z_{-5} + 2Z_{-7}, \\ E_5 &= 2Z_{-2}Z_{-3} + 2Z_{-1}Z_{-4} + 2Z_0Z_{-5} + 2Z_1Z_{-6} + 2Z_{-8}, \\ E_6 &= (Z_{-3})^2 + 2Z_{-2}Z_{-4} + 2Z_{-1}Z_{-5} + 2Z_0Z_{-6} + 2Z_1Z_{-7} + 2Z_{-9}, \\ E_7 &= 2Z_{-3}Z_{-4} + 2Z_{-2}Z_{-5} + 2Z_{-1}Z_{-6} + 2Z_0Z_{-7} + 2Z_1Z_{-8} + 2Z_{-10}, \\ E_8 &= (Z_{-4})^2 + 2Z_{-3}Z_{-5} + 2Z_{-2}Z_{-6} + 2Z_{-1}Z_{-7} + 2Z_0Z_{-8} + 2Z_1Z_{-9} + 2Z_{-11}, \\ E_9 &= 3(Z_0)^2Z_{-1} + 3Z_1(Z_{-1})^2 + 6Z_0Z_1Z_{-2} + 3(Z_{-2})^2 + 3(Z_1)^2Z_{-3} + 6Z_{-1}Z_{-3} \\ &\quad + 6Z_0Z_{-4} + 6Z_1Z_{-5} + 3Z_{-7}, \\ E_{10} &= 3Z_0(Z_{-1})^2 + 3(Z_0)^2Z_{-2} + 6Z_1Z_{-1}Z_{-2} + 6Z_0Z_1Z_{-3} + 6Z_{-2}Z_{-3} + 3(Z_1)^2Z_{-4} \\ &\quad + 6Z_{-1}Z_{-4} + 6Z_0Z_{-5} + 6Z_1Z_{-6} + 3Z_{-8}, \\ E_{11} &= \lambda + (Z_{-1})^3 + 6Z_0Z_{-1}Z_{-2} + 3Z_1(Z_{-2})^2 + 3(Z_0)^2Z_{-3} + 6Z_1Z_{-1}Z_{-3} + 3(Z_{-3})^2 \\ &\quad + 6Z_0Z_1Z_{-4} + 6Z_{-2}Z_{-4} + 3(Z_1)^2Z_{-5} + 6Z_{-1}Z_{-5} + 6Z_0Z_{-6} + 6Z_1Z_{-7} + 3Z_{-9}, \\ E_{12} &= 3(Z_{-1})^2Z_{-2} + 3Z_0(Z_{-2})^2 + 6Z_0Z_{-1}Z_{-3} + 6Z_1Z_{-2}Z_{-3} + 3(Z_0)^2Z_{-4} + 6Z_1Z_{-1}Z_{-4} \\ &\quad + 6Z_{-3}Z_{-4} + 6Z_0Z_1Z_{-5} + 6Z_{-2}Z_{-5} + 3(Z_1)^2Z_{-6} + 6Z_{-1}Z_{-6} + 6Z_0Z_{-7} \\ &\quad + 3Z_{-10} + 6Z_1Z_{-8}, \\ E_{13} &= -\lambda Z_1 + 3Z_{-1}(Z_{-2})^2 + 3(Z_{-1})^2Z_{-3} + 6Z_0Z_{-2}Z_{-3} + 3Z_1(Z_{-3})^2 + 6Z_0Z_{-1}Z_{-4} \\ &\quad + 6Z_1Z_{-2}Z_{-4} + 3(Z_{-4})^2 + 3(Z_0)^2Z_{-5} + 6Z_1Z_{-1}Z_{-5} + 6Z_{-3}Z_{-5} + 6Z_0Z_1Z_{-6} \\ &\quad + 6Z_{-2}Z_{-6} + 3(Z_1)^2Z_{-7} + 6Z_{-1}Z_{-7} + 6Z_0Z_{-8} + 6Z_1Z_{-9} + 3Z_{-11}. \end{aligned}$$

Note that Z_{-11} is the lowest degree coefficient of Z which appears in the E_i 's. It appears in the term $2Z_{-11}$ of E_8 and in the term $3Z_{-11}$ of E_{13} . If $C \in K[y, y^{-1}]((x^{-1}))$ fulfills (b1)–(b6), then

the 13 coefficients $C_1, C_0, C_{-1}, \dots, C_{-11}$ of C , satisfy the equations

$$E_1 = \dots = E_{10} = 0, \quad E_{11} = -y^3, \quad E_{12} = -F_{-4} \quad \text{and} \quad E_{13} = -F_{-5}. \quad (5.20)$$

First we will prove that $F_{-4} = 0$. Assume $F_{-4} \neq 0$. Eliminating in the set of equations

$$E_1 = \dots = E_7 = 0, \quad E_9, E_{10} = 0 \quad \text{and} \quad E_{12} = -F_{-4}$$

the variables $C_0, C_1, C_{-3}, C_{-5}, C_{-6}, C_{-7}, C_{-8}$ and C_{-9} , we obtain

$$C_{-1}^2 C_{-4} = C_{-2}^3 \quad \text{and} \quad 2F_{-4} = 3C_{-1}^2 C_{-2}.$$

Since $F_{-4} = b_1 y + b_{-2} y^{-2}$ and $C_{-1} \in yK[y^{-3}]$ by (b5), necessarily C_{-1} is homogeneous, and so $C_{-1} = c_{-1,1} y$. For the sake of simplicity we write $a := c_{-1,1}$. We set $F_{-3} = y^3$, $C_{-4} := C_{-2}^3 / C_{-1}^2$ and $C_{-2} := 2F_{-4} / 3C_{-1}^2$, and in the set of equations

$$E_1 = \dots = E_7 = 0, \quad E_9 = 0, \quad E_{10} = 0, \quad E_{11} = -F_{-3} \quad \text{and} \quad E_{12} = -F_{-4}$$

we eliminate the variables $C_1, C_{-3}, C_{-5}, C_{-6}, C_{-7}, C_{-8}, C_{-9}$ and C_{-10} . This yields

$$864F_{-4}^2 \lambda = -\frac{256F_{-4}^6}{a^{10}y^{10}} + \frac{864C_0F_{-4}^3}{ay} - 864F_{-4}^2 y^3 + 432a^3 F_{-4}^2 y^3 - 729a^8 C_0^2 y^8,$$

from which we deduce

$$(27a^9 C_0 y^9 - 16F_{-4}^3)^2 = 432a^{10} F_{-4}^2 y^{10} (-2\lambda + (-2 + a^3)y^3).$$

This implies that $-2\lambda + (-2 + a^3)y^3$ is a square in $K((y^{-1}))$, which is only possible if

$$a^3 = 2. \quad (5.21)$$

Now we compute

$$[P, Q] = [P, F] = [x^6, F_{-3}x^{-3}] + [x^6, F_{-4}x^{-4}] + [x^6, F_{-5}x^{-5}] + [2C_1x^4, F_{-3}x^{-3}].$$

Using this, (b2) and the expressions for F_{-3}, F_{-4}, F_{-5} , C_1 and G given in (b2), (b3) and (b6), we obtain

$$6b_1 + 36g_{-2} - \frac{12b_{-2}}{y^3} + \frac{36g_{-5}}{y^3} = 0$$

and

$$-\frac{18g_{-5}^2}{y^8} - \frac{36e_{-10}}{y^8} - \frac{42f_{-7}}{y^8} - \frac{36g_{-2}g_{-5}}{y^5} - \frac{18e_{-7}}{y^5} - \frac{24f_{-4}}{y^5} - \frac{18g_{-2}^2}{y^2} - \frac{6f_{-1}}{y^2} + 18e_{-1}y + 12f_2y = 0.$$

Hence

$$\begin{aligned} f_2 &= -\frac{3e_{-1}}{2}, \quad f_{-1} = -3g_{-2}^2, \quad f_{-4} = -\frac{3}{4}(2g_{-2}g_{-5} + e_{-7}), \quad f_{-7} = -\frac{3}{7}(g_{-5}^2 + 2e_{-10}), \\ b_1 &= -6g_{-2}, \quad b_{-2} = 3g_{-5}. \end{aligned}$$

Now eliminating from the system (5.20) all variables except C_{-1} , we obtain

$$\begin{aligned} R_0 &:= C_{-1}^{10} (3C_{-1}^9 - 36C_{-1}^2 F_{-5}^2 + 18C_{-1}^6 F_{-3} - 96F_{-3}^3 - 6C_{-1}^6 \lambda - 48C_{-1}^3 F_{-3} \lambda - 96F_{-3}^2 \lambda) \\ &\quad - (C_{-1}^6 F_{-5} F_{-4}^2 (-48C_{-1}^3 - 96F_{-3}) + F_{-4}^4 (16C_{-1}^6 + 64C_{-1}^3 F_{-3} + 64F_{-3}^2)) = 0, \end{aligned}$$

and eliminating from the same system all variables except C_{-1} and C_1 , we obtain among others

$$R_1 := 4F_{-4}^2 - C_{-1}^3 (3C_1 C_{-1}^3 + 12F_{-5} + 12C_1 F_{-3}) = 0.$$

Equating to zero the coefficients of R_0 and R_1 , and taking into account (5.21), we obtain the system of equations:

$$\begin{aligned} 0 &= a^3 - 2 \\ 0 &= -\frac{3}{7}(-12(7 + a^3)g_{-5}^2 + a^3(4 + 7a^3)e_{-10}), \end{aligned}$$

$$\begin{aligned}
0 &= -3(-6(-8 + a^3)g_{-2}g_{-5} + a^3(1 + a^3)e_{-7}), \\
0 &= -3(4 + a^3)(-12g_{-2}^2 + a^3e_{-4}), \\
0 &= -3a^3(-2 + a^3)e_{-1}, \\
0 &= -\frac{324}{49}((28 + 14a^3 + a^6)g_{-5}^2 + 2a^6e_{-10})^2, \\
0 &= -\frac{162}{7}(2(-32 - 16a^3 + a^6)g_{-2}g_{-5} + a^6e_{-7})((28 + 14a^3 + a^6)g_{-5}^2 + 2a^6e_{-10}), \\
0 &= -\frac{81}{28}(28a^6(-32 - 16a^3 + a^6)g_{-2}g_{-5}e_{-7} + 7a^{12}e_{-7}^2 \\
&\quad + 4g_{-2}^2(3(3584 + 3584a^3 + 864a^6 - 16a^9 + 5a^{12})g_{-5}^2 + 16a^6(4 + a^3)^2e_{-10})), \\
0 &= -\frac{162}{7}(14(4 + a^3)^2(-32 - 16a^3 + a^6)g_{-2}g_{-5} + 7a^6(4 + a^3)^2g_{-2}^2e_{-7} \\
&\quad + 2a^6e_{-1}((28 + 14a^3 + a^6)g_{-5}^2 + 2a^6e_{-10})), \\
0 &= -81(4(4 + a^3)^4g_{-2}^4 + 2a^6(-32 - 16a^3 + a^6)g_{-2}g_{-5}e_{-1} + a^{12}e_{-1}e_{-7}), \\
0 &= -324a^6(4 + a^3)^2g_{-2}^2e_{-1}, \\
0 &= -3a^{10}(27a^2e_{-1}^2 + 32\lambda + 16a^3\lambda + 2a^6\lambda), \\
0 &= 3a^{10}(-32 + 6a^6 + a^9).
\end{aligned}$$

Eliminating in this system the variables a , e_{-10} , e_{-7} , e_{-4} , e_{-1} and λ , we obtain $g_{-2}^5 = 0$ and $g_{-5}^4 = 0$. So, $F_{-4} = \frac{3g_{-5}}{y^2} - 6g_{-2}y = 0$, as desired.

Now, eliminating from the set of equations $E_1 = \dots = E_{10} = 0$, $E_{12} = 0$ all variables except C_0 and C_{-1} , we obtain $C_0C_{-1}^4 = 0$ (hence $C_0 = 0$), and eliminating from the set of equations $E_1 = \dots = E_{10} = 0$, $E_{11} + F_{-3} = 0$ and $E_{12} = 0$ all variables except C_1 and C_{-1} , we obtain among others

$$8C_{-1}^2F_{-3} = C_{-1}^2(-3C_1^2C_{-1}^2 + 4C_{-1}^3 - 8\lambda),$$

which implies that

$$C_{-1}^2(4C_{-1} - 3C_1^2) = 8(F_{-3} + \lambda) = 8(y^3 + \lambda),$$

because $C_{-1} \neq 0$. Hence C^{-1} is homogeneous, since it belongs to $yK[y^{-3}]$. Write $C_{-1} = ay$. Then

$$3a^2C_1^2y^2 = -8\lambda - 8y^3 + 4a^3y^3.$$

But the right hand side can be only a square in $K((y^{-1}))$ if $a^3 = 2$, and then C_1 is homogeneous with $\deg_y(C_1) = -1$, i.e. $e_{-10} = 0$, which contradicts (b6), since $C_0 = 0$. This rules out the case $\gamma = 2$.

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